ON LOGICAL ANALYSIS OF RELATIVITY THEORIES

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Abstract. The aim of this paper is to give an introduction to our axiomatic logical analysis of relativity theories.

1. introduction

Our general aim is to build up relativity theories as theories in the sense of mathematical logic. So we axiomatize relativity theories within pure first-order logic (FOL) using simple, comprehensible and transparent basic assumptions (axioms). We strive to prove all the surprising predictions of relativity from a minimal number of convincing axioms. We eliminate tacit assumptions from relativity by replacing them with explicit axioms (in the spirit of the foundation of mathematics and Tarski's axiomatization of geometry). We also elaborate logical and conceptual analysis of our theories.

Logical axiomatization of physics, especially that of relativity theory, is not a new idea, among others, it goes back to such leading scientists as Hilbert, Reichenbach, Carnap, Gödel, and Tarski. Relativity theory was intimately connected to logic from the beginning, it was one of the central subjects of logical positivism. For a short survey on the broader literature, see, e.g., [2]. Our aims go beyond these approaches in that along with axiomatizing relativity theories we also analyze in detail their logical and conceptual structure and, in general, investigate them in various ways (using our logical framework as a starting point).

A novelty in our approach is that we try to keep the transition from special relativity to general relativity logically transparent and illuminating. We "derive" the axioms of general relativity from those of special relativity in two natural steps. First we extend our axiom system for special relativity with accelerated observers (sec.7). Then we eliminate the distinguished status of inertial observers at the level of axioms (sec.8).

Some of the questions we study to clarify the logical structure of relativity theories are:

- What is believed and why?
- Which axioms are responsible for certain predictions?

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- What happens if we discard some axioms?
- Can we change the axioms and at what price?

Our aims stated in the first paragraph reflect, partly, the fact that we axiomatize a physical theory. Namely, in physics the role of axioms (the role of statements that we assume without proofs) is more fundamental than in mathematics. Among others, this is why we aim to formulate simple, logically transparent and intuitively convincing axioms.Our goal is that on our approach, surprising or unusual predictions be theorems and not assumed as axioms. For example, the prediction "no faster than light motion ..." is a theorem on our approach and not an axiom, see Thm.5.1.

Getting rid of unnecessary axioms is especially important in a physical theory. When we check the applicability of a physical theory in a situation, we have to check whether the axioms of the theory hold or not. For this we often use empirical facts (outcomes of concrete experiments). However, these correspond to existentially quantified theorems[1] rather than to universally quantified statements—which the axioms usually are. Thus while we can easily disprove the axioms by referring to empirical facts, we can verify these axioms only to a certain degree. Some of the literature uses the term 'empirical fact' for universal generalization of an empirical fact elevated to the level of axioms, see, e.g., [16, §4], [28]. We simply call these generalizations (empirical) axioms.

2. WHY RELATIVITY?

For one thing, Einstein's theory of relativity not just had but still has a great impact on many areas of science. It has also greatly affected several areas in the philosophy of science. Relativity theory has an impact even on our every day life, e.g., via GPS technology (which cannot work without relativity theory). Any theory with such an impact is also interesting from the point of view of axiomatic foundations and logical analysis.

Since spacetime is a similar geometrical object as space, axiomatization of relativity theories (or spacetime theories in general) is a natural continuation of the works of Euclid, Hilbert, Tarski and many others axiomatizing the geometry of space.

3. WHY AXIOMATIC METHOD?

There are many examples showing the benefits of using axiomatic method. For example, if we decompose relativity theories into little parts (axioms), we can check what happens to our theory if we drop, weaken or replace an axiom or we can take any prediction, such as

[1]We do not want to assume every experimental fact as an axiom. We only want them to be consequences of our theories.

the twin paradox, and check which axiom is and which is not needed to derive it. This kind of reverse thinking helps to answer the why-type questions. For details on answering why-type questions by the methodology of the present work, see [1, 12–13.], [30].

The success story of axiomatic method in the foundations of mathematics also suggests that it is worth applying this method in the foundations of spacetime theories [13], [14]. Let us note here that Euclid's axiomatic-deductive approach to geometry also made a great impression on the young Einstein, see [18].

Among others, logical analysis makes relativity theory modular: we can change some axioms, and our logical machinery ensures that we can continue working in the modified theory. This modularity might come handy, e.g., when we want to unify general relativity and quantum theory to a theory of quantum gravity. For further reasons why to apply the axiomatic method to spacetime theories, see, e.g., [2], [1], [17], [26], [27].

4. Why first-order logic?

We aim to provide a logical foundation for spacetime theories similar to the rather successful foundations of mathematics, which, for good reasons, was performed strictly within FOL. One of these reasons is that FOL helps to avoid tacit assumptions. Another is that FOL has a complete inference system while second-order logic (or higher-order logic) cannot have one.

Still another reason for choosing FOL is that it can be viewed as a fragment of natural language with unambiguous syntax and semantics. Being a *fragment of natural language* is useful in our project because one of our aims is to make relativity theory accessible to a broad audience. *Unambiguous syntax and semantics* are important, because they make it possible for the reader to always know what is stated and what is not stated by the axioms. Therefore they can use the axioms without being familiar with all the tacit assumptions and rules of thumb of physics (which one usually learns via many, many years of practice).

For further reasons why to stay within FOL when dealing with axiomatic foundations, see, e.g., [1, §Appendix: Why FOL?], [7], [29, §11], [33], [34].

5. Special relativity

Before we present our axiom system let us go back to Einstein's original (logically non-formalized) postulates. Einstein based his special theory of relativity on two postulates, the principle of relativity and the light principle: "The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of coordinates in uniform translatory motion." and "Any ray of light moves in the 'stationary'

system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body.", see [11].

The logical formulation of Einstein's principle of relativity is not an easy task since it is difficult to capture axiomatically what "the laws of nature" are in general. Nevertheless, the principle of relativity can be captured by our FOL approach, see [1], [20, §2.8.3].

Instead of formulating the two original principles, we formulate the following consequence of theirs: "the speed of light signals is the same in every direction everywhere according to every inertial observer" (and not just according to the 'stationary' observer). Here we will base our axiomatization on this consequence and call it light axiom. We will soon see that the light axiom can be regarded as the key assumption of special relativity.

Since we want to axiomatize special relativity, we have to fix some formal language in which we will write up our axioms. Let us see the basic concepts (the "vocabulary" of the FOL language) we will use. We would like to speak about motion. So we need a basic concept of things that can move. We will call these object *bodies*.[2] The light axiom requires a distinguished type of bodies called *photons* or *light signals*.[3] We will represent motion as the changing of spatial location in time. Thus we will use reference frames for coordinatizing events (meetings of bodies). Time and space will be marked by *quantities*. The structure of quantities will be an *ordered field* in place of the field of real numbers.[4] For simplicity, we will associate special bodies to reference frames. These special bodies will be called "observers." Observations will be formalized/represented by means of the *worldview relation*.

To formalize the ideas above, let us fix a natural number $d \geq 2$ for the dimension of spacetime. To axiomatize theories of the d-dimensional spacetime, we will use the following two-sorted FOL language:

$$\{\, B, \mathsf{IOb}, \mathsf{Ph},\ Q, +, \cdot,\ \mathsf{W} \,\},$$

[2]By bodies we mean anything which can move, e.g., test-particles, reference frames, electromagnetic waves, etc.

[3]Here we use light signals and photons as synonyms because it is not important here whether we think of them as particles or electromagnetic waves. The only thing that matters here is that they are "things that can move." So they are bodies in the sense of our FOL language.

[4] Using ordered fields in place of the field of real numbers increases the flexibility of the theory and reduces the amount of mathematical presuppositions. For further motivation in this direction, see, e.g., [7]. Similar remarks apply to our other flexibility-oriented decisions, e.g., to treat the dimension of spacetime as a variable.

where B (bodies) and Q (quantities) are the two sorts,[5] IOb (inertial observers) and Ph (light signals or photons) are one-place relation symbols of sort B, $+$ and $\cdot$ are two-place function symbols of sort Q, and W (the worldview relation) is a $2+d$-place relation symbol the first two arguments of which are of sort B and the rest are of sort Q.

Atomic formulas $\mathsf{IOb}(k)$ and $\mathsf{Ph}(p)$ are translated as "*k is an inertial observer*," and "*p is a photon*," respectively. To speak about coordinatization, we translate $\mathsf{W}(k, b, x_1, \ldots, x_{d-1}, t)$ as "*body k coordinatizes body b at space-time location $\langle x_1, \ldots, x_{d-1}, t\rangle$*," (i.e., at space location $\langle x, \ldots, x_{d-1}\rangle$ and at instant t). Sometimes we use the more picturesque expressions *sees* or *observes* for *coordinatizes.* However, these cases of "seeing" and "observing" have nothing to do with visual seeing or observing; they only mean associating coordinate points to bodies.

The above, together with statements of the form $x = y$ are the so-called *atomic formulas* of our FOL language, where x and y can be arbitrary variables of the same sort, or terms built up from variables of sort Q by using the two-place operations $\cdot$ and $+$. The *formulas* are built up from these atomic formulas by using the logical connectives *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$), *if-and-only-if* ($\leftrightarrow$) and the quantifiers *exists* ($\exists$) and *for all* ($\forall$). For the precise definition of the syntax and semantics of FOL, see, e.g., [9, §1.3].

To meaningfully formulate the light axiom, we have to provide some algebraic structure for the quantities. Therefore, in our first axiom, we state some usual properties of addition $+$ and multiplication $\cdot$ true for real numbers.

> <u>AxFd</u>**:** The quantity part $\langle Q, +, \cdot\rangle$ is a Euclidean field, i.e.,
> - $\langle Q, +, \cdot\rangle$ is a field in the sense of abstract algebra,
> - the relation $\leq$ defined by $x \leq y \stackrel{d}{\Longleftrightarrow} \exists z\ x + z^2 = y$ is a linear ordering on Q, and
> - Positive elements have square roots: $\forall x\ \exists y\ x = y^2 \vee -x = y^2$.

The field-axioms (see, e.g., [9, 40–41.]) say that $+$, $\cdot$ are associative and commutative, they have neutral elements 0, 1 and inverses $-$, $/$ respectively, with the exception that 0 does not have an inverse with respect to $\cdot$, as well as $\cdot$ is additive with respect to $+$. We will use 0, 1, $-$, $/$, $\sqrt{\ }$ as derived (i.e., defined) operation symbols.

AxFd is a "mathematical" axiom in spirit. However, it has physical (even empirical) relevance. Its physical relevance is that we can add and multiply the outcomes of our measurements and some basic rules apply to these operations. Physicists usually use all properties of the real numbers tacitly, without stating explicitly which property is assumed and why. The two properties of real numbers which are the most

[5] That our theory is two-sorted means only that there are two types of basic objects (bodies and quantities) as opposed to, e.g., set theory where there is only one type of basic objects (sets).

difficult to defend from an empirical point of view are the Archimedean property, see [24], [25, §3.1], and the supremum property,[6] see the remark after the introduction of axiom Cont on p.13.

Euclidean fields got their name after their role in Tarski's FOL axiomatization of Euclidean geometry [32]. By AxFd we can reason about the Euclidean structure of a coordinate system the usual way, we can introduce Euclidean distance, speak about straight lines, etc. In particular, we will use the following notation for $\bar{x}, \bar{y} \in Q^n$ (i.e., $\bar{x}$ and $\bar{y}$ are n-tuples over Q) if $n \geq 1$:

$$|\bar{x}| \stackrel{d}{=} \sqrt{x_1^2 + \cdots + x_n^2}, \quad \text{and} \quad \bar{x} - \bar{y} \stackrel{d}{=} \langle x_1 - y_1, \ldots, x_n - y_n \rangle.$$

We will also use the following two notations:

$$\bar{x}_s \stackrel{d}{=} \langle x_1, \ldots, x_{d-1} \rangle \quad \text{and} \quad x_t \stackrel{d}{=} x_d$$

for the *space component* and the *time component* of $\bar{x} = \langle x_1, \ldots, x_d \rangle \in Q^d$, respectively.

Now let us see how the light axiom can be formalized in our FOL language.

<u>AxPh</u>**:** For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction. Formally:

$$\forall m \ \exists c_m \ \forall \bar{x}\bar{y} \ \mathsf{IOb}(m) \rightarrow$$
$$\big(\exists p \ \ \mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y})\big) \leftrightarrow |\bar{y}_s - \bar{x}_s| = c_m \cdot |y_t - x_t|.$$

Axiom AxPh has an immediate physical meaning. This axiom is not only implied by the two original principles of relativity, but it is well supported by experiments, such as the Michelson-Morley experiment. Moreover, it has been continuously tested ever since then. Nowadays it is tested by GPS technology.

Axiom AxPh says that "It is *possible* for a photon to move from $\bar{x}$ to $\bar{y}$ iff ...". So, a notion of possibility plays a role here. In the present paper we work in an extensional framework, as is customary in geometry and in spacetime theory. However, it would be more natural to treat this "possibility phenomenon" in a modal logic framework, and this is more emphatically so for relativistic dynamics [4]. It would be interesting to explore the use of modal logic in our logical analysis of relativity theory. This investigation would be a nice unification of the works of Imre Ruzsa's school on modal logic and the works of our Tarskian spirited school on axiomatic foundations of relativity theory.

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[6]The supremum property (i.e., *every* nonempty and bounded *subset* of the real numbers has a least upper bound) implies the Archimedean property. So if we want to get ourselves free from the Archimedean property, we have to leave this property, too.

Robin Hirsch's work can be considered as a first step along this road [19].

Let us note that AxPh does not require that the speed of light be the same for every inertial observer or that it be nonzero. It requires only that the speed of light according to a fixed inertial observer be a quantity which does not depend on the direction or the location.

Why do we not require that the speed of light is nonzero? The main reason is that we are building our logical foundation of spacetime theories examining thoroughly each part of each axiom to see where and why we should assume them. Another (more technical) reason is that it will be more natural to include this assumption ($c_m \neq 0$) in our auxiliary axiom AxSm on page 8.

Our next axiom connects the worldviews of different inertial observers by saying that all observers observe the same "external" reality (the same set of events). Intuitively, by the event occurring for m at $\bar{x}$, we mean the set of bodies m observes at $\bar{x}$. Formally:

$$\mathsf{ev}_m(\bar{x}) \stackrel{d}{=} \{b : \mathsf{W}(m, b, \bar{x})\}.$$

<u>AxEv</u>**:** All inertial observers coordinatize the same set of events:

$$\forall mk \ \ \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \ \ \rightarrow \ \ \forall \bar{x} \ \exists \bar{y} \ \forall b \ \ \mathsf{W}(m, b, \bar{x}) \leftrightarrow \mathsf{W}(k, b, \bar{y}).$$

This axiom is very natural and tacitly assumed in the non-axiomatic approaches to special relativity, too.

Basically we are done. We have formalized the light axiom AxPh. We have introduced two supporting axioms (AxFd and AxEv) for the light axiom which are simple and natural; however, we cannot simply omit them without loosing some of the meaning of AxPh. The field axiom enables us to speak about distances, time differences, speeds, etc. The event axiom ensures that different inertial observers see the same events.

In principle, we do not need more axioms for analyzing/axiomatizing special relativity, but let us introduce two more simplifying ones. We could leave them out without loosing the essence of our theory, it is just that the formalizations of the theorems would become more complicated.

<u>AxSf</u>**:** Any inertial observer sees himself on the time axis:

$$\forall m \ \ \mathsf{IOb}(m) \rightarrow \ \ \big(\forall \bar{x} \ \ \mathsf{W}(m, m, \bar{x}) \leftrightarrow x_1 = 0 \wedge x_2 = 0 \wedge x_3 = 0\big).$$

The role of AxSf is nothing more than making it easier to speak about the motion of reference frames via the motion of their time axes. Identifying the motion of reference frames with the motion of their time axes is a standard simplification in the literature. AxSf is a way to formally capture this simplifying identification.

Our last axiom is a symmetry axiom saying that all inertial observers use the same units of measurements.

AxSm: Any two inertial observers agree about the spatial distance between two events if these two events are simultaneous for both of them; furthermore, the speed of light is 1:

$$\forall mk\ \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \to \forall \bar{x}\bar{y}\bar{x}'\bar{y}'\ x_t = y_t \wedge x'_t = y'_t \wedge \\ \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}') \wedge \mathsf{ev}_m(\bar{y}) = \mathsf{ev}_k(\bar{y}') \to |\bar{x}_s - \bar{y}_s| = |\bar{x}'_s - \bar{y}'_s|, \text{ and}$$

$$\forall m\ \mathsf{IOb}(m)\ \to\ \exists p\ \mathsf{Ph}(p) \wedge \mathsf{W}(m,p,0,0,0,0) \wedge \mathsf{W}(m,p,1,0,0,1).$$

Let us see how AxSm states that "all inertial observers use the same units of measurements." That "the speed of light is 1" (besides that the speed of light is nonzero) means only that observers are using units measuring time distances compatible with the units measuring spatial distances, such as light years or light seconds. The first part of AxSm means that different observers use the same unit measuring spatial distances. This is so because if two events are simultaneous for both observers, they can measure their spatial distance and the outcome of their measurements are the same iff the two observers are using the same units to measure spatial distances.
Our axiom system for special relativity contains these 5 axioms only:

$$\mathsf{SpecRel} \stackrel{d}{=} \{\mathsf{AxFd}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSf}, \mathsf{AxSm}\}.$$

In an axiom system, the axioms are the "price" we pay, and the theorems are the "goods" we get for them. Therefore, we strive for putting only simple, transparent, easy-to-believe statements in our axiom systems. We want to get all the hard-to-believe predictions as theorems. For example, we prove from SpecRel that it is impossible for inertial observers to move faster than light relative to each other ("No FTL travel" for science fiction fans). In the following, $\vdash$ means logical derivability.

Theorem 5.1. (no faster than light inertial observers)

$$\mathsf{SpecRel} \vdash \forall mk\bar{x}\bar{y}\quad \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \\ \wedge \mathsf{W}(m,k,\bar{x}) \wedge \mathsf{W}(m,k,\bar{y}) \wedge \bar{x} \neq \bar{y}\ \to\ |\bar{y}_s - \bar{x}_s| < |y_t - x_t|.$$

For a geometrical proof of Thm.5.1, see [6].

In relativity theory we are often interested in comparing the worldviews of different observers. So we introduce the worldview transformation between observers m and k as the following binary relation:

$$\mathsf{w}_{mk}(\bar{x},\bar{y})\ \stackrel{d}{\iff}\ \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y}).$$

By Thm.5.2, the worldview transformations between inertial observers in the models of SpecRel are Poincaré transformations, i.e., transformations which preserve the so-called Minkowski-distance $(y_t - x_t)^2 - |\bar{y}_s - \bar{x}_s|^2$ of d-tuples $\bar{y}, \bar{x}$. For the definition, we refer to [10, 110.] or [23, 66–69.].

Theorem 5.2.

$$\mathsf{SpecRel} \vdash \forall m,k\ \ \mathsf{IOb}(m)\wedge\mathsf{IOb}(k)\ \rightarrow\ \mathsf{w}_{mk} \text{ is a Poincaré transformation.}$$

For the proof of Thm.5.2, see [3, Thm.11.10, 640.] or [29, Thm.3.2.2, 22.]. By Thm.5.2, all predictions of special relativity, such as "moving clocks slow down," are provable from SpecRel. For details, see, e.g., [2, §1], [3, §2], [1, §2.5].

6. LOGICAL ANALYSIS

Let us illustrate here by a simple example what we mean by logical analysis of a theory. In AxEv we have assumed that all observers see the same (possibly infinite) meetings of bodies. Let us try to weaken AxEv to an axiom assuming something similar but only for finite meetings of bodies. A natural candidate is one of the following finite approximations of AxEv:

$\underline{\mathsf{AxMeet_n}}$**:** All inertial observers see the same n-meetings of bodies:

$$\forall mkb_1\dots b_n\bar{x}\ \ \mathsf{IOb}(m)\wedge\mathsf{IOb}(k)\wedge\mathsf{W}(m,b_1,\bar{x})\wedge\dots\wedge\mathsf{W}(m,b_n,\bar{x})$$
$$\rightarrow\ \exists\bar{y}\ \ \mathsf{W}(k,b_1,\bar{y})\wedge\dots\wedge\mathsf{W}(k,b_n,\bar{y}).$$

For example, $\mathsf{AxMeet_1}$ means only that inertial observers see the same bodies. Let us also introduce axiom scheme Meet_ω as the collection of all the axioms $\mathsf{AxMeet_n}$. By Prop.6.1, $\mathsf{AxMeet_n}$ is strictly weaker assumption than $\mathsf{AxMeet_{n+1}}$ and AxEv is strictly stronger than all the axioms of Meet_ω together.

Proposition 6.1.

$$\mathsf{AxEv}\ \vdash\ \mathsf{AxMeet_{n+1}}\vdash\mathsf{AxMeet_n} \tag{1}$$
$$\mathsf{AxMeet_n}\ \not\vdash\ \mathsf{AxMeet_{n+1}} \tag{2}$$
$$\mathsf{Meet}_\omega\ \not\vdash\ \mathsf{AxEv} \tag{3}$$

Proof. Item (1) follows easily by the formulations of the axioms.

To prove Item (2), we are going to construct a model of $\mathsf{AxMeet_n}$ in which $\mathsf{AxMeet_{n+1}}$ is not valid. Let $B=\{b_i : i\le n\}$.Let all the bodies be inertial observers. Let b_0 see all the bodies in $\langle 0,\dots,0\rangle$ and none of them in any other coordinate points, i.e., let $\mathsf{W}(b_0,b_i,\bar{x})$ hold iff $\bar{x}=\langle 0,\dots,0\rangle$; and for all $k\neq 0$ let b_k see all the bodies but b_i at coordinate points $\langle i,\dots,i\rangle$ for all $i\le n$, i.e., let $\mathsf{W}(b_k,b_i,\bar{x})$ hold iff $\bar{x}=\langle j,\dots,j\rangle$ and $i\neq j$. In this model, all inertial observers see all the possible n-meetings. So $\mathsf{AxMeet_n}$ is valid in this model. However, the only inertial observer who sees the $n+1$-meeting $\{b_0,\dots,b_n\}$ is b_0. So $\mathsf{AxMeet_{n+1}}$ is not valid in this model.

We are going to prove Item (3) by a similar model construction. The only difference is that now Q will be infinite. For simplicity, let Q be the set of natural numbers. Let all the other parts of the model

be defined in the same way. Now all the inertial observers see all the possible n-meetings of the bodies for all natural numbers n. So $\mathsf{AxMeet_n}$ is valid in this model for all natural number n. Hence Meet_ω is valid in this model. However, only b_0 sees the event $\{b_1, b_2, \ldots,\}$. So AxEv is not valid in this model. $\square$

Now we will use that there are no stationary (i.e., motionless) light signals. So let us formalize this statement.

$\underline{\mathsf{Ax(c \neq 0)}}$**:** Inertial observers do not see stationary light signals.

$$\forall m p \bar{x} \bar{y} \quad \mathsf{IOb}(m) \land \mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y}) \land x_t \neq y_t \ \rightarrow \ \bar{x}_s \neq \bar{y}_s.$$

Proposition 6.2.

$$\mathsf{AxMeet_3, AxFd, AxPh, Ax(c \neq 0)} \vdash \mathsf{AxEv} \tag{4}$$

$$\mathsf{AxMeet_2, AxFd, AxPh, Ax(c \neq 0)} \nvdash \mathsf{AxEv} \tag{5}$$

$$\mathsf{Meet}_\omega, \mathsf{AxFd, AxPh} \nvdash \mathsf{AxEv} \tag{6}$$

Proof. First let us make some general observations. By AxFd, there is no nondegenerate triangle in Q^d whose sides are of slope c. This is clear if $c = 0$; and in the case $c \neq 0$, this can be shown by contradiction using the fact that the vertical projection of a triangle of this kind is a triangle whose one side is the sum of the other two sides. Therefore, AxFd and AxPh together imply that any inertial observer m sees the events in which a particular photon participates on a line of slope c_m.

By AxFd, AxPh and $\mathsf{Ax(c \neq 0)}$, every inertial observer m sees different meetings of photons at different coordinate points. This is so since (by AxFd) for every pair of points there is a line of slope $c_m \neq 0$ containing only one of the points. Hence, by AxPh, there is a photon seen by m only at one of the two coordinate points.

Let us now prove Item (4). Let m and k be inertial observers and let $\bar{x}$ be a coordinate point. To prove AxEv, we have to find a coordinate point $\bar{x}'$ such that $\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}')$. To find this $\bar{x}'$, let $\bar{y} = \langle x_1 + c_m, x_2, \ldots, x_{d-1}, x_t + 1\rangle$, $\bar{z} = \langle x_1 - c_m, x_2, \ldots, x_{d-1}, x_t + 1\rangle$ and $\bar{w} = \langle x_1, \ldots, x_{d-1}, x_t + 2\rangle$, see Fig.1.

By AxPh, there are photons p_1, p_2 and p_3 such that $p_1, p_2 \in \mathsf{ev}_m(\bar{x})$, $p_2, p_3 \in \mathsf{ev}_m(\bar{y})$, $p_1 \in \mathsf{ev}_m(\bar{z})$ and $p_3 \in \mathsf{ev}_m(\bar{w})$. Since m sees every photon on a line of slope c_m, he sees the meeting of p_1 and p_2 only at $\bar{x}$ and does not see the meeting of p_1 and p_3.

Since $\mathsf{AxMeet_3}$ implies $\mathsf{AxMeet_2}$, k sees the same meetings of pairs of photons. So there is a $\bar{x}'$ where k sees p_1 and p_2 meet. $\bar{x}'$ is the only point where k sees both p_1 and p_2. This is so because k sees different meetings of photons at different points but sees the same 3-meetings as m. So if there were another point, say $\bar{x}''$, where k sees p_1 and p_2, there were photons $p' \in ev_k(\bar{x}')$ and $p'' \in \mathsf{ev}_k(\bar{x}'')$ such that $p' \notin ev_k(\bar{x}'')$, $p'' \notin \mathsf{ev}_k(\bar{x}')$ and k does not see the meeting of p' and p''. By axiom $\mathsf{AxMeet_3}$ m has to see the meetings $\{p_1, p_2, p'\}$ and $\{p_1, p_2, p''\}$. The

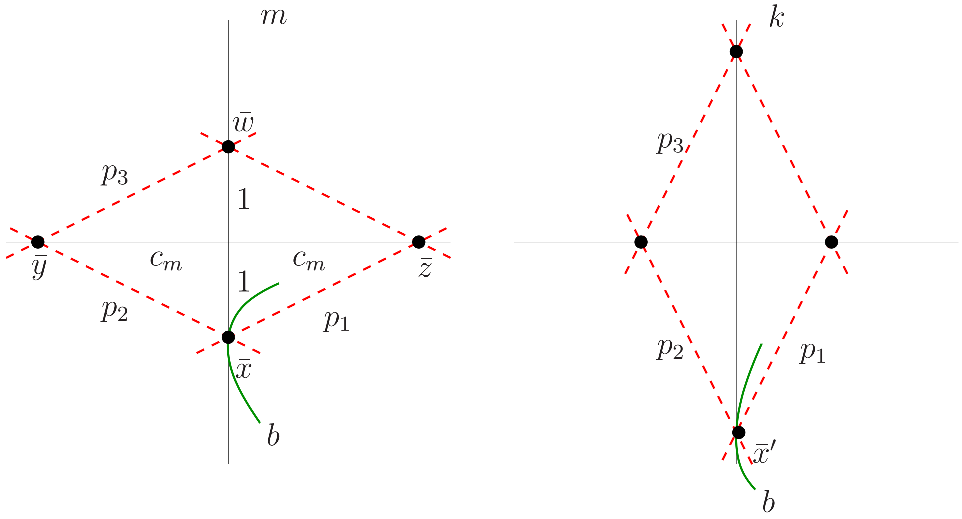

FIGURE 1.

only point where m can see these meetings is $\bar{x}$ since $\bar{x}$ the only point where m sees p_1 and p_2 meet. Therefore m sees the meeting of p' and p'' at $\bar{x}$. Thus, by AxMeet_3, k also has to see the meeting of p' and p'', but k does not see it. Hence $\bar{x}'$ is the only point where k sees both p_1 and p_2.

Let b be a body such that $\mathsf{W}(m, b, \bar{x})$. By AxMeet_3, k has to see the meeting of p_1, p_2 and b. This point has to be $\bar{x}'$ since the only point where p_1 and p_2 meet is $\bar{x}'$. Since b was an arbitrary body, we have $\mathsf{ev}_m(\bar{x}) \subseteq \mathsf{ev}_k(\bar{x}')$. The same argument shows that $\mathsf{ev}_k(\bar{x}') \subseteq \mathsf{ev}_m(\bar{x})$. So $\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}')$ as desired.

We are going to prove Item (5), by constructing a model. Let $\langle Q, +, \cdot \rangle$ be the field of real numbers. Let us denote the set of natural numbers by ω. Let $B = \{m, k\} \cup \{b_i : i \in \omega\} \cup \{p : p \text{ is a line of slope } 1\}$. Let m and k be all the inertial observers and let the lines of slope 1 be all the photons. Let m and k see the photon p at coordinate point $\bar{x}$ iff $\bar{x} \in p$. Let m see all the bodies b_i at $\bar{x}$ iff $x_t = 0$. Let k see all the bodies $b_0, \ldots, b_n, \ldots$ but b_i at $\bar{x}$ iff $x_t = i$ (i.e., iff $\bar{x}$ is in the horizontal hyperplane $\{\bar{y} \in Q^d : y_t = i\}$).[7] It is straightforward from this construction that axioms AxFd, AxPh and $\mathsf{Ax(c \neq 0)}$ are valid in this model. Since every line of slope 1 intersects every horizontal hyperplane, m and k see the same 2-meetings of bodies. Hence AxMeet_2 is also valid in this model. However, the only inertial observer who sees the meeting $\{b_i : i \in \omega\}$ is m. So AxEv is not valid in this model.

We prove Item (6) by a similar construction. The only difference is that now the set of bodies is $B = \{m, k\} \cup \{b_i : i \in \omega\} \cup \{p :$

[7] If $d = 2$, vertical lines can be used instead of horizontal hyperplanes, which gives a counterexample with bodies having more natural properties.

p is a vertical line$\}$; and the photons are the vertical lines. It is straightforward from the construction that axioms AxFd, AxPh are valid in this model ($c = 0$). Since every vertical line intersects every horizontal hyperplane, m and k see the same n-meetings of bodies. Hence Meet_ω is also valid in this model. However, only m sees the meeting $\{b_i : i \in \omega\}$. So AxEv is not valid in this model. □

Prop.6.2 shows that a price to weaken axiom AxEv to AxMeet_3 is to assume that there are no stationary light signals. Since AxSm contains this assumption, we can simply replace AxEv with AxMeet_3 in $\mathsf{SpecRel}$. A natural continuation of this investigation can be a search for assumptions that allow us to weaken AxMeet_3 to AxMeet_2. A possible candidate is that bodies move along straight lines and the dimension d is at least 3. The proof of Item (5) shows that assuming only that bodies move along straight lines is not enough, if $d = 2$.

We have several similar investigations on the logical connections of axioms and predictions, see, e.g., [4], [29, §5] on dynamics, [21], [29, §4,§7], [31] on twin paradox, [1] on kinematics, time-dilation and length-contraction, twin paradox, etc.

7. ACCELERATED OBSERVERS

In $\mathsf{SpecRel}$ we restricted our attention to inertial observers. It is a natural idea to generalize the theory by including accelerated observers as well. It is explained in the classic textbook [23, 163–165.] that the study of accelerated observers is a natural first step (from special relativity) towards general relativity.

We have not introduced the concept of observers as a basic one because it can be defined as follows: an *observer* is nothing other than a body who "observes" (coordinatizes) some other bodies somewhere, this property can be captured by the following formula of our language:

$$\mathsf{Ob}(m) \stackrel{d}{\iff} \exists b\bar{x}\ \mathsf{W}(m, b, \bar{x}).$$

Our key axiom about accelerated observers is the following:

AxCmv: At each moment of his life, every accelerated observer sees (coordinatizes) the nearby world for a short while in the same way as an inertial observer does.

For formulation of AxCmv in our FOL language, see [21], [29] or [6].

Axiom AxCmv ties the behavior of accelerated observers to those of inertial ones. Justification of this axiom is given by experiments. We call two observers *co-moving* at an event if they "see the nearby world for a short while in the same way" at the event. By this notion AxCmv says that at each event of an observer's life, he has a co-moving inertial observer. We can think of a dropped spacepod as a co-moving inertial observer of an accelerated spaceship (at the event of dropping). Or,

if a spaceship switches off its engines, it will move on as a co-moving inertial spaceship would.

Our next two axioms ensure that the worldviews of accelerated observers are big enough. They are generalized versions of the corresponding axioms for inertial observers, but now postulated for all observers.

<u>AxEv^-</u>**:** If m sees k in an event, then k cannot deny it:

$$\forall m, k \in \mathsf{Ob}\ \ \mathsf{W}(m, k, \bar{x}) \to \exists \bar{y}\ \ \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y}).$$

<u>AxSf^-</u>**:** Any observer sees himself in an interval of the time axis:

$$\forall m \in \mathsf{Ob}\ \forall \bar{x}\ \ \mathsf{W}(m, m, \bar{x})\ \to\ x_1 = x_2 = x_3 = 0 \quad \text{and}$$
$$\forall \bar{x}\bar{y}\quad \mathsf{W}(m, m, \bar{y}) \wedge \mathsf{W}(m, m, \bar{x})\ \to\ \forall t\ \ x_t < t < y_t \to \mathsf{W}(m, m, 0, 0, 0, t).$$

Our last two axioms will ensure that the worldlines of accelerated observers are "tame" enough, e.g., they have velocities at each moment. In SpecRel, the worldview transformations between inertial observers are affine maps, the next axiom will state that the worldview transformations between accelerated observers are approximately affine, wherever they are defined.

<u>AxDf</u>**:** The worldview transformations have linear approximations at each point of their domain (i.e., they are differentiable).

For a precise formalization of AxDf, see, e.g., [6].

We note that AxDf implies that the worldview transformations are functions with open domains. However, if the numberline has gaps, still there can be crazy motions. Our last assumption is an axiom scheme supplementing AxDf by excluding these gaps.

<u>Cont</u>**:** Every definable, bounded and nonempty subset of Q has a supremum (i.e., least upper bound).

In Cont "definable" means "definable in the language of AccRel, parametrically." For a precise formulation of Cont, see [21, 692.] or [29, §10.1]. Cont is a "mathematical axiom" in spirit. It is Tarski's FOL version of Hilbert's continuity axiom in his axiomatization of geometry, see [15, 61–162.], fitted to the language of AccRel. When Q is the field of real numbers, Cont is automatically true.

That Cont requires the existence of supremum only for sets definable in the language of AccRel instead of every set, is important not only because by this trick we can keep our theory within FOL (which is crucial in a foundational work), but also because it makes this postulate closer to the the physical/empirical level. The latter is true because Cont does not speak about "any fancy subset" of the quantities, just those "physically meaningful" sets which can be defined in the language of our (physical) theory.

Adding this 5 axioms to SpecRel, we get an axiom system for accelerated observers:

$$\mathsf{AccRel} \stackrel{d}{=} \mathsf{SpecRel} \cup \{\mathsf{AxCmv}, \mathsf{AxEv}^-, \mathsf{AxSf}^-, \mathsf{AxDf}\} \cup \mathsf{Cont}.$$

As an example we show that the so-called *twin paradox* can be naturally formulated and analyzed logically in AccRel. Our axiomatic approach also makes it possible to analyze the details of the twin paradox (e.g., who sees what, when) with the clarity of logic, see [1, 139–150.] for part of such an analysis.

According to the twin paradox, if a twin makes a journey into space (accelerates), he will return to find that he has aged less than his twin brother who stayed at home (did not accelerate). We formulate the twin paradox in our FOL language as follows.

TwP: Every inertial observer m measures at least as much time as any other observer k between any two events e_1 and e_2 in which they meet; and they measure the same time iff they have encountered the very same events between e_1 and e_2:

$$\forall m \in \mathsf{IOb}\ \forall k \in \mathsf{Ob}\ \forall \bar{x}\bar{x}'\bar{y}\bar{y}'\ \ x_t < y_t \wedge x'_t < y'_t \wedge$$
$$m, k \in \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}') \wedge m, k \in \mathsf{ev}_m(\bar{y}) = \mathsf{ev}_k(\bar{y}')\ \rightarrow\ y'_t - x'_t \le y_t - x_t$$
$$\wedge \big(y'_t - x'_t = y_t - x_t \leftrightarrow enc_m(\bar{x},\bar{y}) = enc_k(\bar{y}',\bar{y}')\big),$$

where $enc_m(\bar{x},\bar{y}) = \{\mathsf{ev}_m(\bar{z}) : \mathsf{W}(m,m,\bar{z}) \wedge x_t \le z_t \le y_t\}$.

Theorem 7.1.

$$\mathsf{AccRel} \vdash \mathsf{TwP} \tag{7}$$
$$\mathsf{AccRel} - \mathsf{AxDf} \vdash \mathsf{TwP} \tag{8}$$
$$\mathsf{AccRel} - \mathsf{Cont} \nvdash \mathsf{TwP} \tag{9}$$
$$\mathsf{Th}(\mathbb{R}) \cup \mathsf{AccRel} - \mathsf{Cont} \nvdash \mathsf{TwP} \tag{10}$$

For the proof of Thm.7.1, see [21] or [29, §7].

Item (10) of Thm.7.1 states that Cont cannot be replaced with the whole FOL theory of real numbers in AccRel if we do not want to loose TwP from its consequences.

Our theory AccRel is also strong enough to predict the gravitational time-dilation effect of general relativity via Einstein's equivalence principle, see [22], [29].

8. GENERAL RELATIVITY

Our theory of accelerated observers AccRel speaks about two kinds of observers, inertial and accelerated ones. Some axioms are postulated for inertial observers only, some apply to all observers. We get an axiom system GenRel for general relativity by stating the axioms of AccRel in a generalized form in which they are postulated for all observers, inertial and accelerated ones equally. In other words, we will change all axioms of AccRel in the same spirit as AxSf^- and AxEv^- were obtained from AxSf and AxEv, respectively. This kind of change $\mathsf{AccRel} \mapsto \mathsf{GenRel}$ can be regarded as a "democratic revolution" with the slogan "all observers should be equivalent, the same laws should apply to all of them." Here

"law" translates as "axiom." This idea originates with Einstein (see his book [12, Part II, ch.18]).

For simplicity, we will use an equivalent version of the symmetry axiom AxSm (see [1, Thm.2.8.17(ii), 138.] or [29, Thm.3.1.4, 21.]), and we will require the speed of photons to be 1 in AxPh^- (as opposed to requiring it in AxSm^-).

<u>AxPh^-</u>**:** The velocity of photons an observer "meets" is 1 when they meet, and it is possible to send out a photon in each direction where the observer stands.

<u>AxSm^-</u>**:** Meeting observers see each other's clocks slow down with the same rate.

For a precise formulation of these axioms, see [6], [29].

We introduce an axiom system for general relativity as the collection of the following axioms:

$$\mathsf{GenRel} \stackrel{d}{=} \{\mathsf{AxFd}, \mathsf{AxPh}^-, \mathsf{AxEv}^-, \mathsf{AxSf}^-, \mathsf{AxSm}^-, \mathsf{AxDf}\} \cup \mathsf{Cont}.$$

Axiom system GenRel contains basically the same axioms as $\mathsf{SpecRel}$, the difference is that they are assumed only locally but for all the observers.

Thm.8.1 below states that the models of GenRel are exactly the spacetimes of usual general relativity. For the notion of a Lorentzian manifold we refer to [10, 55.], [23, 241.] and [3, sec.3.2].

Theorem 8.1 (Completeness theorem)**.** GenRel *is complete with respect to its standard models, i.e., with respect to Lorentzian Manifolds over real closed fields.*

This theorem can be regarded as a completeness theorem in the following sense. Let us consider Lorentzian manifolds as intended models of GenRel. How can we do that? We give a method for constructing a model of GenRel from each Lorentzian manifold; and conversely, we show that each model of GenRel is obtained this way from a Lorentzian manifold. After this is elaborated, we have defined what we mean by a formula φ in the language of GenRel being valid in a Lorentzian manifold. Then completeness means that for any formula φ in the language of GenRel, we have $\mathsf{GenRel} \vdash \varphi$ iff φ is valid in all Lorentzian manifolds over real closed fields. This is completely analogous to the way in which Minkowskian spacetimes were regarded as intended models of $\mathsf{SpecRel}$ in the completeness theorem of $\mathsf{SpecRel}$, see [3, Thm.11.28, 681.] and [20, §4].

We call the worldline of an observer *timelike geodesic*, if each of its points has a neighborhood within which this observer "maximizes measured time (wrist-watch time)" between any two encountered events. For formalization of this concept in our FOL language, see, e.g., [6].

According to the definition above, if there are only a few observers, then it is not a big deal that a worldline is a time-like geodesic (it is easy

to be maximal if there are only a few to be compared to). To generate a real competition for the rank of having a timelike geodesic worldline, we postulate the existence of many observers by the following axiom scheme of comprehension.

Compr: For any parametrically definable timelike curve in any observers worldview, there is another observer whose worldline is the range of this curve.

A precise formulation of Compr can be obtained from that of its variant in [3, 679.].

An axiom schema Compr guarantees that our definition of a geodesic coincides with that in the literature on Lorentzian manifolds. Therefore we also introduce the following theory:

$$\mathsf{GenRel}^{+} \stackrel{d}{=} \mathsf{GenRel} \cup \mathsf{Compr}.$$

So in our theory GenRel^{+}, our concept of timelike geodesic coincides with the standard concept in the literature on general relativity. All the other key concepts of general relativity, such as curvature or Riemannian tensor field, are definable from timelike geodesics. Therefore we can treat all these concepts (including the concept of metric tensor field) in our theory GenRel^{+} in a natural way.

In general relativity, Einstein's field equations (EFE) provide the connection between the geometry of spacetime and the energy-matter distribution (given by the energy-momentum tensor field). Since in GenRel^{+} all the geometric concepts of spacetime are definable, we can use Einstein's equation as a definition of the energy-momentum tensor, see, e.g., [8] or [10, §13.1, 169.], or we can extend the language of GenRel^{+} with the concept of energy-momentum tensor and assume Einstein's equations as axioms. As long as we do not assume anything more of the energy-momentum tensor than its connection to the geometry described by Einstein's equations, there is no real difference in these two approaches. In both approaches, we can add extra conditions about the energy-momentum tensor to our theory, e.g., the dominant energy condition or, e.g., that the spacetimes are vacuum solutions.

9. CAN PHYSICS GIVE FEEDBACK TO LOGIC?

There is observational evidence suggesting that in our physical universe there exist regions supporting potential non-Turing computations. Namely, it is possible to design a physical device in relativistic spacetime which can compute a non-Turing computable task, e.g., which can decide whether ZF set theory is consistent. This empirical evidence is making the theory of hypercomputation more interesting and gives new challenges to the physical Church Thesis, see, e.g., [5].

These new challenges do more than simply providing a further connection between logic and spacetime theories; they also motivate the need for logical understanding of spacetime theories.

10. CONCLUDING REMARKS

We have axiomatized both special and general relativity in FOL. Moreover, via our theory AccRel, we have axiomatized general relativity so that each of its axioms can be traced back to its roots in the axioms of special relativity. Axiomatization is not our final goal. It is merely an important first step toward logical and conceptual analysis. We are only at the beginning of our ambitious project.[1]

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